

CASTELNUOVO-MUMFORD REGULARITY AND RATLIFF-RUSH CLOSURE

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ABSTRACT. We establish relationships between the Castelnuovo-Mumford regularity of standard graded algebras and the Ratliff-Rush closure of ideals. These relationships can be used to compute the Ratliff-closure and the regularities of the Rees algebra and the fiber ring. As a consequence, these regularities are equal for large classes of monomial ideals in two variables, thereby confirming a conjecture of Eisenbud and Ulrich for these cases.

Dedicated to Giuseppe Valla on the occasion of his seventieth anniversary

INTRODUCTION

Let R be a standard graded algebra over a commutative ring with unity. Let $H_{R_+}^i(R)$ denote the i -th local cohomology module of R with respect to the graded ideal R_+ of elements of positive degree and set $a_i(R) = \max\{n \mid H_{R_+}^i(R)_n \neq 0\}$ with the convention $a_i(R) = -\infty$ if $H_{R_+}^i(R) = 0$. The Castelnuovo-Mumford regularity is defined by

$$\operatorname{reg} R := \max\{a_i(R) + i \mid i \geq 0\}.$$

It is well known that $\operatorname{reg} R$ controls many important invariants of the graded structure of R (see e.g. [1], [6], [30]).

The motivation for our work originates from the following conjecture of Eisenbud and Ulrich [7, Conjecture 1.3].

Conjecture. Let A be a standard graded algebra over a field k . Let \mathfrak{m} be the maximal graded ideal of A and I a homogeneous \mathfrak{m} -primary ideal which is generated by forms of the same degree. Then $\operatorname{reg} R(I) = \operatorname{reg} F(I)$, where $R(I) = \bigoplus_{n \geq 0} I^n$ is the Rees algebra and $F(I) = \bigoplus_{n \geq 0} I^n / \mathfrak{m} I^n$ is the fiber ring of I .

In general, it is very difficult to estimate $\operatorname{reg} R(I)$ because $R(I)$ is a standard graded algebra over A . On the other hand, as $F(I)$ is a standard graded algebra over k , $\operatorname{reg} F(I)$ can be effectively computed in terms of a minimal free resolution.

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Note that if d is the degree of the generators of I , then $F(I) \cong k[I_d]$, the subalgebra of A generated by the elements of I_d .

Using the characterization of $\text{reg } R(I)$ by means of a superficial sequence, the authors were able to settle the above conjecture in the affirmative when I is an ideal in $k[x, y]$ generated by a set of monomials in degree d which contains $x^d, x^{d-1}y, y^d$. The solution suggests that both $\text{reg } R(I)$ and $\text{reg } F(I)$ are related to the behavior of the Ratliff-Rush filtration. Inspired by this finding, this paper will study the relationships between the Castelnuovo-Mumford regularity and the Ratliff-Rush closure.

Let (A, \mathfrak{m}) be an arbitrary local ring and I an arbitrary ideal of A . Recall that the Ratliff-Rush closure of I is defined as the ideal

$$\tilde{I} = \bigcup_{n \geq 1} I^{n+1} : I^n.$$

It is a refinement of the integral closure of I and $\tilde{I} = I$ if I is integrally closed. If I is a regular ideal, i.e. if I contains non-zerodivisors, \tilde{I} is the largest ideal sharing the same higher powers with I [20]. In particular, the Ratliff-Rush filtration $\tilde{I}^n, n \geq 0$, carries important information on the blowups, the associated graded ring of I , and the Hilbert function of an \mathfrak{m} -primary ideal I (see e.g. [10], [11], [14], [24]).

In general, the computation of \tilde{I} is hard because $I^{n+1} : I^n = I^n : I^{n-1}$ does not imply $I^{n+2} : I^{n+1} = I^{n+1} : I^n$. We call the least integer $m \geq 0$ such that $\tilde{I} = I^{n+1} : I^n$ for all $n \geq m$ the Ratliff-Rush index of I and denote it by $s(I)$. If we know an upper bound for $s(I)$, we can easily compute \tilde{I} . For an \mathfrak{m} -primary ideal I , Elias [8] already gave a bound for $s(I)$ in terms of the postulation numbers of I and of ideals of the form $I/(x)$, where x belongs to a given superficial sequence of I . For an arbitrary ideal I , we will show that $I^{n+1} : I = I^n$ for $n \geq \text{reg } R(I)$. From this it follows that

$$s(I) \leq \max\{\text{reg } R(I) - 1, 0\}.$$

Since there are various bound for $\text{reg } R(I)$ in terms of other well known invariants of I ([3], [4], [5], [17], [18], [23], [25], [31]), the above bound for $s(I)$ provides us a practical tool to compute \tilde{I} .

A remarkable feature of the Ratliff-Rush closure is the property that $\tilde{I}^n = I^n$ for all n sufficiently large if I is a regular ideal. Again, if $\tilde{I}^n = I^n$, then it does not necessarily imply $\widetilde{I^{n+1}} = I^{n+1}$. We call the least integer $m \geq 1$ such that $\tilde{I}^n = I^n$ for all $n \geq m$ the Ratliff-Rush regularity of the ideal I and denote it by $s^*(I)$. We will show that $\tilde{I}^n = I^{n+t} : I^t$ for $t \geq \text{reg } R(I) - n$. From this it immediately follows that

$$s^*(I) \leq \max\{\text{reg } R(I), 1\}.$$

Using the strong result that $\text{reg } R(I) = \text{reg } G(I)$, where $G(I)$ denotes the associated graded ring of I , one can also deduce this bound from the bound $s^*(I) \leq \max\{a_1(G(I)) + 1, 1\}$ given by Puthenpurakal in [19].

Now one may ask whether $s^*(I)$ can be used to estimate $\text{reg } R(I)$. If A is a two-dimensional Buchsbaum local ring with depth $A > 0$ (e. g. if A is Cohen-Macaulay)

and I is an \mathfrak{m} -primary ideal, which is not a parameter ideal, we show that

$$\operatorname{reg} R(I) = \max\{r_J(I), s^*(I)\},$$

where J is an arbitrary minimal reduction of I and $r_J(I)$ denotes the reduction number of I with respect to J . As an application we give a negative answer to a question of Rossi and Swanson [21, Section 4] which asks whether $s^*(I) \leq r_J(I)$ always holds. In fact, if the answer were yes, this would imply $r_J(I) = \operatorname{reg} R(I)$ independent of the choice of J . However, Huckaba [15] already showed that $r_J(I)$ may depend on the choice of J .

Our interest in Buchsbaum rings comes from the fact that the conjecture of Eisenbud and Ulrich is not true if one does not put further assumption on the standard graded algebra A . We shall see that if the conjecture were true for factor rings of A , then A must be a Buchsbaum ring. If A is an one-dimensional Buchsbaum ring, we will show that $\operatorname{reg} R(I) = \operatorname{reg} F(I)$ always holds. If A is a two-dimensional Buchsbaum ring with depth $A > 0$ and I is not a parameter ideal, we show that

$$\operatorname{reg} F(I) = \max\{r_J(I), s_{\text{in}}^*(I)\}.$$

Here, $s_{\text{in}}^*(I)$ denotes the least integer $m \geq 1$ such that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$ for all $n \geq m$, where d is the degree of the generators of I . Since nd is the initial degree of \tilde{I}^n , we call $s_{\text{in}}^*(I)$ the initial Ratliff-Rush regularity of I . The above formulas establish unexpected relationships between the Castelnuovo-Mumford regularity and the Ratliff-Rush closure, which can be used to compare $\operatorname{reg} R(I)$ and $\operatorname{reg} F(I)$.

If I is an \mathfrak{m} -primary monomial ideal in a polynomial ring $k[x, y]$ which is generated by forms of degree d , then I has a natural minimal reduction $J = (x^d, y^d)$. Using the above formulas we are able to show that $\operatorname{reg} R(I) = \operatorname{reg} F(I)$ in the following cases:

- (1) $I = (x^d, y^d) + (x^{d-i}y^i \mid a \leq i \leq b)$, where $a \leq b \leq d$ are given positive integers.
- (2) $x^d, x^{d-1}y, y^d \in I$.

These large classes of ideals indicate that the conjecture of Eisenbud and Ulrich may be true for polynomial rings over a field. In fact, Ulrich communicated to the last author that he and Eisenbud always thought of a polynomial ring in their conjecture.

Note that the equality $\operatorname{reg} R(I) = \operatorname{reg} F(I)$ was already studied by Cortadellas and Zarzuela [2], and Jayanthan and Nanduri [9] for an ideal I in a local ring. However, their results are too specific to be recalled here.

The paper is divided into four sections. In Section 1 we recall basic results on the Castelnuovo-Mumford regularity of the Rees algebra. The bound $s(I) \leq \max\{\operatorname{reg} R(I) - 1, 0\}$ will be proved in this section. In Section 2 we study the relationship between $s^*(I)$ and $\operatorname{reg} R(I)$ and prove the bound $s^*(I) \leq \max\{\operatorname{reg} R(I), 1\}$ and the formula $\operatorname{reg} R(I) = \max\{r_J(I), s^*(I)\}$. In Section 3 we investigate the conjecture of Eisenbud and Ulrich and prove the formula $\operatorname{reg} k[I_d] = \max\{r_J(I), s_0^*(I)\}$. In Section 4 we apply our approach to monomial ideals in two variables and settle the conjecture of Eisenbud and Ulrich in the afore mentioned cases.

1. REGULARITY OF THE REES ALGEBRA

Let (A, \mathfrak{m}) be a local ring with $\dim A > 0$ and I an ideal of A . Let $R(I) = \bigoplus_{n \geq 0} I^n$ be the Rees algebra of I . We shall see that $\text{reg } R(I)$ can be characterized in terms of a superficial sequence which generates a reduction of I . Without loss of generality we may assume that the residue field of A is infinite.

An element $x \in I$ is called *superficial* for I if there is an integer c such that

$$(I^{n+1} : x) \cap I^c = I^n$$

for all large n . A system of elements x_1, \dots, x_s in I is called a *superficial sequence* of I if x_i is a superficial element of I in $A/(x_1, \dots, x_{i-1})$, $i = 1, \dots, s$. Superficial sequences can be characterized by means of filter-regular sequences in the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

A system of homogeneous elements z_1, \dots, z_s in $G(I)$ is called *filter-regular* if

$$[(z_1, \dots, z_{i-1}) : z_i]_n = (z_1, \dots, z_{i-1})_n,$$

for sufficiently large n , $i = 1, \dots, s$. It is easy to see that z_1, \dots, z_s is filter-regular if and only if $x_i \notin P$ for all associated primes $P \not\supseteq G(I)_+$ of (z_1, \dots, z_{i-1}) , $i = 1, \dots, s$ (see [28]). This characterization is especially useful in finding filter-regular sequences.

For every element $x \in I$ we denote by x^* the residue class of x in I/I^2 .

Lemma 1.1. [29, Lemma 6.2] *x_1, \dots, x_s is a superficial sequence of I if and only if x_1^*, \dots, x_s^* forms a filter-regular sequence of $G(I)$.*

Note that the condition $x_i \notin (x_1, \dots, x_{i-1}) + I^2$, $i = 1, \dots, s$, in [29, Lemma 6.2] is superfluous.

An ideal $J \subseteq I$ is called a *reduction* of I if there exists an integer n such that $I^{n+1} = JI^n$. The least integer n with this property is called the *reduction number* of I with respect to J . We will denote it by $r_J(I)$. A reduction is minimal if it is minimal with respect to containment.

The following relationship between minimal reductions and superficial sequences is more or less known. For completeness we include a proof here.

Lemma 1.2. *Every minimal reduction J of I can be generated by a superficial sequence of I .*

Proof. Let Q denote the ideal in $G(I)$ generated by the elements x^* , $x \in J$. Then Q is generated by $(J + I^2)/I^2$. Since $I^{n+1} = JI^n$, $Q_{n+1} = G(I)_{n+1}$. Therefore, $Q \not\subseteq P$ for any prime $P \not\supseteq G(I)_+$. Using prime avoidance we can find a filter-regular sequence $z_1, \dots, z_s \in G(I)$ such that $Q = (z_1, \dots, z_s)$. Choose $x_i \in J$ such that $x_i^* = z_i$, $i = 1, \dots, s$. Then $(x_1, \dots, x_s) + I^2 = J + I^2$. Hence

$$(x_1, \dots, x_s)I^n + I^{n+2} = JI^n + I^{n+2} = I^{n+1}.$$

By Nakayama's Lemma, this implies $I^{n+1} = (x_1, \dots, x_s)I^n$. Therefore, (x_1, \dots, x_s) is a reduction of I . By the minimality of J , we must have $J = (x_1, \dots, x_s)$. The conclusion now follows from Lemma 1.1. \square

One can characterize $\text{reg } R(I)$ in terms of a superficial sequence that generates a reduction of I . The following characterization is a reformulation of [29, Theorem 4.8], where it is assumed that x_1^*, \dots, x_s^* is a filter-regular sequence.

Theorem 1.3. *Let x_1, \dots, x_s be a superficial sequence of I such that $J = (x_1, \dots, x_s)$ is a reduction of I . Then*

$$\begin{aligned} \text{reg } R(I) &= \text{reg } G(I) \\ &= \min \{n \geq r_J(I) \mid I^{n+1} \cap [(x_1, \dots, x_{i-1}) : x_i] = (x_1, \dots, x_{i-1})I^n, i = 1, \dots, s\}. \end{aligned}$$

Theorem 1.3 will play a crucial role in our paper. One can use it to compute $\text{reg } F(I)$ in terms of J as we shall see later.

Superficial elements are related to the regularity by the following property, which is a reformulation of [29, Lemma 4.4 (i)].

Lemma 1.4. *Let x be a superficial element of I . Then $I^{n+1} \cap (x) = xI^n$ for $n \geq \text{reg } R(I)$.*

This lemma led us to the following property of colon ideals of powers of I .

Proposition 1.5. *Let I be a regular ideal. Then $I^{n+1} : I = I^n$ for $n \geq \text{reg } R(I)$.*

Proof. It is well-known that if I is a regular ideal, every superficial element of I is a non-zerodivisor (see e.g. [22, Lemma 1.2]). Therefore, $0 : x = 0$ if x is a superficial element for I . By Lemma 1.4, $I^{n+1} : x = I^n + (0 : x) = I^n$ for $n \geq \text{reg } R(I)$. Since $I^n \subseteq I^{n+1} : I \subseteq I^{n+1} : x$, this implies $I^{n+1} : I = I^n$ for $n \geq \text{reg } R(I)$. \square

Remark 1.6. Actually, the proof of [29, Lemma 4.4 (i)] shows more, namely that $I^{n+1} \cap (x) = xI^n$ for $n \geq \max\{a_0(G(I)), a_1(G(I)) + 1\}$. If I is a regular ideal, $a_0(G(I)) < a_1(G(I))$ [13, Theorem 5.2]. Therefore, $I^{n+1} : I = I^n$ for $n \geq a_1(G(I)) + 1$. By the definition of the Castelnuovo-Mumford regularity and Theorem 1.3, we always have $a_1(G(I)) + 1 \leq \text{reg } G(I) = \text{reg } R(I)$.

Recall that the *Ratliff-Rush closure* of I is defined as the ideal

$$\tilde{I} = \bigcup_{n \geq 1} I^{n+1} : I^n.$$

In general, the computation of \tilde{I} is hard because $I^{n+1} : I^n = I^n : I^{n-1}$ does not imply $I^{n+2} : I^{n+1} = I^{n+1} : I^n$ [21]. Therefore, it is of great interest to have an upper bound for the least integer $m \geq 0$ such that $\tilde{I} = I^{n+1} : I^n$ for all $n \geq m$. We call this integer the *Ratliff-Rush index* of I and denote it by $s(I)$. Note that $s(I) = 0$ means $\tilde{I} = I$.

Theorem 1.7. *Let I be a regular ideal. Then $s(I) \leq \max\{\text{reg } R(I) - 1, 0\}$.*

Proof. Applying Proposition 1.5 we have

$$I^{n+1} : I^n = (I^{n+1} : I) : I^{n-1} = I^n : I^{n-1}$$

for $n \geq \text{reg } R(I)$. Thus, $I^{n+1} : I^n$ is the same ideal for all $n \geq \text{reg } R(I) - 1$ and equals \tilde{I} . \square

There are plenty examples with $s(I) = 0$ and $\text{reg } R(I)$ arbitrarily large. For instance, take $I = \mathfrak{m}$. It is clear that $s(\mathfrak{m}) = 0$. Since $\text{reg } R(\mathfrak{m}) \geq r_J(\mathfrak{m})$ for any minimal reduction J of \mathfrak{m} , one can easily construct local rings such that $\text{reg } R(\mathfrak{m})$ is arbitrarily large.

According to Theorem 1.7, if $\text{reg } R(I) \leq c$ for some integer $c \geq 0$, then $\tilde{I} = I^{c+1} : I^c$. So one can compute \tilde{I} if one knows an upper bound for $\text{reg } R(I)$. There have been several works giving upper bounds for $\text{reg } R(I)$ ([3], [4], [5], [17], [18], [23], [25], [31]). We consider here only a general upper bound for $\text{reg } R(I)$ in terms of the extended degree.

Let I be an \mathfrak{m} -primary ideal. Following [3] and [17] we call a numerical function $D(I, M)$ an *extended degree* of a finitely generated A -module M with respect to I if the following conditions are satisfied:

- (i) $D(I, M) = D(I, M/L) + \ell(L)$, where L is the largest submodule of M of finite length,
- (ii) $D(I, M) \geq D(I, M/xM)$ for a generic element x in I ,
- (iii) $D(I, M) = e(I, M)$ if M is a Cohen-Macaulay module, where $e(I, M)$ denotes the multiplicity of M with respect to I .

We refer the readers to [3], [17], [31] for several kinds of extended degrees. For $M = A$ we simply use the notations $D(I)$ and $e(I)$ instead of $D(I, R)$ and $e(I, R)$.

Theorem 1.8. *Let A be a d -dimensional ring with $\text{depth } A > 0$. Let I be an \mathfrak{m} -primary ideal. Set $c(I) = D(I) - e(I)$, where $e(I)$ denotes the multiplicity of I .*

- (i) *If $d = 1$, then $s(I) \leq e(I) + c(I) - 1$,*
- (ii) *If $d \geq 2$, then $s(I) \leq e(I)^{(d-1)!-1} [e(I)^2 + e(I)c(I) + 2c(I) - e(I)]^{(d-1)!} - c(I)$.*

Proof. The assertion follows from Theorem 1.7 and [23, Theorem 3.3], where the right sides of the bounds were shown to be upper bounds for $\text{reg } R(I)$. Note that [23, Theorem 3.3] was proved for the case $I = \mathfrak{m}$. However, the proof can be extended to an arbitrary \mathfrak{m} -primary ideal I . It was carried out in [17, Theorem 4.4], where a more compact but weaker bound for $\text{reg } R(I)$ is given. \square

Corollary 1.9. *Let A be a d -dimensional Cohen-Macaulay ring. Let I be an \mathfrak{m} -primary ideal.*

- (i) *If $d = 1$, then $s(I) \leq e(I) - 1$,*
- (ii) *If $d \geq 2$, then $s(I) \leq e(I)^{2(d-1)!-1} [e(I) - 1]^{(d-1)!}$.*

Similar upper bounds for $s(I)$ were already given by Elias in [8, Theorem 2.1], which is slightly worse than Corollary 1.9(ii) in the case $d \geq 2$. His proof involves the postulation numbers of a set of quotient ideals $I/(x)$, where x is an element of a given superficial sequence of I generating a minimal reduction of I .

2. RATLIFF-RUSH FILTRATION

Let (A, \mathfrak{m}) be a local ring with $\dim A > 0$ and I an ideal of A . One call the sequence of ideals \tilde{I}^n , $n \geq 1$, the *Ratcliff-Rush filtration* with respect to I . It is well

known that for $n \geq 1$,

$$\tilde{I}^n = \bigcup_{t \geq 0} I^{n+t} : I^t$$

and, if I is a regular ideal, $\tilde{I}^n = I^n$ for n sufficiently large [20].

We call the least integer $m \geq 1$ such that $\tilde{I}^n = I^n$ for $n \geq m$ the *Ratliff-Rush regularity* of I and denote it by $s^*(I)$. Note that $\tilde{I}^n = I^n$ does not necessarily imply $\tilde{I}^{n+1} = I^{n+1}$ (see e.g. [21]).

In this section we shall see that $s^*(I)$ is strongly related to $\text{reg } R(I)$.

Proposition 2.1. *Let I be a regular ideal. Then*

- (i) $\tilde{I}^n = I^{n+t} : I^t$ for all $t \geq \text{reg } R(I) - n$,
- (ii) $s^*(I) \leq \max\{\text{reg } R(I), 1\}$.

Proof. By Proposition 1.5 we have $I^{n+1} : I = I^n$ for $n \geq \text{reg } R(I)$. Therefore,

$$I^{n+t+1} : I^{t+1} = (I^{n+t+1} : I) : I^t = I^{n+t} : I^t$$

for $t \geq \text{reg } R(I) - n$, which proves (i). If $n \geq \text{reg } R(I)$, we can put $t = 0$ in (i). Hence $\tilde{I}^n = I^n : I^0 = I^n$, which proves (ii). \square

As pointed out in Remark 1.6, we can replace $\text{reg } R(I)$ by $a_1(G(I)) + 1$ in Proposition 2.1. So we can recover the bound $s^*(I) \leq \max\{a_1(G(I)) + 1, 1\}$ proved by Puthenpurakal in [19, Theorem 4.3]. Note that Puthenpurakal considers the least integer $m \geq 0$ such that $\tilde{I}^n = I^n$ for $n \geq m$, whereas we require $m \geq 1$ because one always has $\tilde{I}^0 = I^0 = A$. On the other hand, we can also deduce Proposition 2.1 (ii) from Puthenpurakal's result by using the inequality $a_1(G(I)) + 1 \leq \text{reg } G(I) = \text{reg } R(I)$.

Proposition 2.1 has the interesting consequence that if c is an upper bound for $\text{reg } R(I)$, then $\tilde{I}^n = I^c : I^{c-n}$ for $n < c$ and $\tilde{I}^n = I^n$ for $n \geq c$. In particular, if $\text{reg } R(I) \leq 1$, then $s^*(I) = 1$, i.e. $\tilde{I}^n = I^n$ for all $n \geq 1$. We will use this fact to give a large class of ideals with $s^*(I) = 1$.

Recall that a system of elements x_1, \dots, x_r in A is a *d-sequence* if the following two conditions are satisfied:

- (i) x_i is not contained in the ideal generated by the rest of the system, $i = 1, \dots, r$,
- (ii) $(x_1, \dots, x_i) : x_{i+1}x_k = (x_1, \dots, x_i) : x_{i+1}$ for all $i = 0, \dots, r-1$ and $k = i+1, \dots, r$.

This notion was introduced by Huneke in [16]. Examples of *d-sequences* are abundant such as the maximal minors of an $r \times (r+1)$ generic matrix and systems of parameters in Buchsbaum rings. It was showed in [29, Corollary 5.7] that $\text{reg } R(I) = 0$ if and only if I is generated by a *d-sequence*. Therefore, Proposition 2.1 (ii) implies the following result.

Corollary 2.2. *Let I be a regular ideal generated by a d-sequence. Then $\tilde{I}^n = I^n$ for all $n \geq 1$.*

It is also known that $\tilde{I}^n = I^n$ for all $n \geq 1$ if and only if $G(I)$ contains a non-zero-divisor [11, (1.2)]. This fact can be used to find examples with $s^*(I) = 1$ and $\text{reg } R(I)$ arbitrarily large.

Example 2.3. Let $R = k[X]/P$, where $k[X]$ is a polynomial ring and P is a homogeneous prime generated by forms of any given degree d . Let A be the localization of R at its maximal graded ideal. Then $G(\mathfrak{m}) \cong R$. Since $\text{depth } R > 0$, $s^*(\mathfrak{m}) = 1$. By Theorem 1.3, $\text{reg } R(\mathfrak{m}) = \text{reg } G(\mathfrak{m}) = \text{reg } R$. It is known that $\text{reg } R + 1$ is an upper bound for the degree of the generators of P [6]. Thus, $\text{reg } R(\mathfrak{m}) \geq d - 1$.

Despite the possible large difference between $\text{reg } R(I)$ and $s^*(I)$ we can use $s^*(I)$ to characterize $\text{reg } R(I)$ in the following case.

Recall that A is called a *Buchsbaum ring* if every system of parameters x_1, \dots, x_r of A is a *weak sequence*, i.e.

$$(x_1, \dots, x_{i-1}) : x_i = (x_1, \dots, x_{i-1}) : \mathfrak{m}$$

for $i = 1, \dots, r$. Huneke showed that A is a Buchsbaum ring if and only if every system of parameters forms a d -sequence [16, Proposition 1.7]. Therefore, $\text{reg } R(I) = 0$ and $s^*(I) = 1$ if I is a parameter ideal in a Buchsbaum ring. If I is not a parameter ideal, we have the following formula for $\text{reg } R(I)$.

Theorem 2.4. *Let A be a two-dimensional Buchsbaum ring with $\text{depth } A > 0$. Let I be an \mathfrak{m} -primary ideal, which is not a parameter ideal. Then*

$$\text{reg } R(I) = \max\{r_J(I), s^*(I)\} = \min\{n \geq r_J(I) \mid \tilde{I}^n = I^n\}.$$

where J is an arbitrary minimal reduction of I .

Proof. By Theorem 1.3, $\text{reg } R(I) \geq r_J(I)$. Since I is not a parameter ideal, $r_J(I) \geq 1$. Hence, $\text{reg } R(I) \geq 1$. By Proposition 2.1 (ii), this implies $\text{reg } R(I) \geq s^*(I)$. Thus, $\text{reg } R(I) \geq \max\{r_J(I), s^*(I)\}$. Since

$$\max\{r_J(I), s^*(I)\} \geq \min\{n \geq r_J(I) \mid \tilde{I}^n = I^n\},$$

it suffices to show that $\text{reg } R(I) \leq \min\{n \geq r_J(I) \mid \tilde{I}^n = I^n\}$.

By Lemma 1.2, there is a superficial sequence x, y of I such that $J = (x, y)$. Since $\text{depth } A > 0$, I is a regular ideal. Hence $0 : x = 0$. By Theorem 1.3, this implies

$$\text{reg } R(I) = \min\{n \geq r_J(I) \mid I^{n+1} \cap [(x) : y] = xI^n\}.$$

We will show that $I^{n+1} \cap [(x) : y] = I^{n+1} \cap (x)$ for $n \geq r_J(I)$. Let f be an arbitrary element of $I^{n+1} \cap [(x) : y]$. Since $I^{n+1} = (x, y)I^n$, there are elements $g, h \in I^n$ such that $f = gx + hy$. Since $fy \in (x)$, $h \in (x) : y^2$. Since x, y is a d -sequence, $(x) : y^2 = (x) : y$. Hence $hy \in y[(x) : y^2] \subseteq y[(x) : y] \subseteq (x)$. This implies $f \in (x)$. So we can conclude that $I^{n+1} \cap [(x) : y] \subseteq I^{n+1} \cap (x)$. Since the converse inclusion is obvious, $I^{n+1} \cap [(x) : y] = I^{n+1} \cap (x)$. Therefore,

$$\begin{aligned} \text{reg } R(I) &= \min\{n \geq r_J(I) \mid I^{n+1} \cap (x) = xI^n\} \\ &= \min\{n \geq r_J(I) \mid I^{n+1} : x = I^n\}. \end{aligned}$$

Note that $I^n \subseteq I^{n+1} : x \subseteq \widetilde{I^{n+1}} : x = \widetilde{I}^n$ by [22, Lemma 3.1 (5)]. If $\widetilde{I}^n = I^n$, this implies $I^{n+1} : x = I^n$. Thus, $\text{reg } R(I) \leq \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\}$. \square

The formula $\text{reg } R(I) = \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\}$ provides us a practical way to compute $\text{reg } R(I)$ because we only need to check the condition $\widetilde{I}^n = I^n$ successively for $n \geq r_J(I)$. Moreover, comparing to Theorem 1.3, we do not need a superficial sequence which generates a reduction of I . To find such a sequence is in general not easy.

It is known that the reduction numbers may be different for different minimal reductions [15]. Since the reduction number is very useful in the study of local rings (see e.g. [31]), it is of great interest to know when $r_J(I)$ is independent of the choice of J . We can use Theorem 2.4 to give a sufficient condition for the invariance of the reduction numbers.

Corollary 2.5. *Let A be a two-dimensional Buchsbaum ring with depth $A > 0$ and I an \mathfrak{m} -primary ideal. If there exists a minimal reduction J of I such that $s^*(I) < r_J(I)$, then the reduction numbers of all minimal reductions of I equal $\text{reg } R(I)$.*

Proof. Since $r_J(I) > 1$, I is not a parameter ideal. By Theorem 2.4, the assumption implies $\text{reg } R(I) = r_J(I)$. If there is a minimal reduction J' of I with $r_{J'}(I) \neq r_J(I)$, from the formula $\text{reg } R(I) = \max\{r_{J'}(I), s^*(I)\}$ we can deduce that $\text{reg } R(I) = s^*(I)$, a contradiction. \square

There are plenty ideals with $s^*(I) = 1$ and $r_J(I)$ arbitrarily large. For instance, take Example 2.3, where A is chosen to be a two-dimensional Buchsbaum ring. Then $r_J(\mathfrak{m}) = \text{reg } R(\mathfrak{m})$ by Theorem 2.4. As we have seen there, $s^*(\mathfrak{m}) = 1$ while $\text{reg } R(\mathfrak{m})$ can be arbitrarily large. We shall see in the last section that there are examples such that $s^*(I) = r_J(I)$ for a minimal reduction J , but $s^*(I) > r_{J'}(I)$ for another minimal reduction J' of I .

Let $br(I)$ denote the big reduction number of I which is defined by

$$br(I) = \max\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

To estimate the big reduction number is usually a hard problem. If $s^*(I) = r_J(I)$ for some minimal reduction J of I , we can deduce from Theorem 2.4 that $br(I) = r_J(I)$. Under the assumption of Theorem 2.4, we could not find any example with $br(I) < s^*(I)$. So we conjecture that $br(I) \geq s^*(I)$ in this case.

In the following we will give an alternative formula for $\text{reg } R(I)$, which involves only the Ratliff-Rush closure of a power of I . This formula is based on the following observation.

Lemma 2.6. *Let A be a two-dimensional Buchsbaum ring with depth $A > 0$ and I an \mathfrak{m} -primary ideal. Let J be an arbitrary minimal reduction of I . Let x, y be a superficial sequence of I such that $J = (x, y)$. Set $r = r_J(I)$. For $n \geq r$, we have*

$$I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x).$$

Proof. The case $n = r$ is trivial. Let f be an arbitrary element of $I^{n+1} : x$, $n \geq r+1$. Since $I^{n+1} = (x, y)I^n$, there are elements $g, h \in I^n$ such that $xf = xg + yh$. From this it follows that $h \in I^n \cap [(x) : y]$. As showed in the proof of Theorem 2.4,

$$I^n \cap [(x) : y] = I^n \cap (x) = x(I^n : x).$$

Hence $h = xh'$ for some element $h' \in I^n : x$. Thus, $xf = xg + xyh'$. Since x is a non-zerodivisor, $f = g + yh' \in I^n + y(I^n : x)$. So we have $I^{n+1} : x \subseteq I^n + y(I^n : x)$. Since the inverse inclusion is obvious, we can conclude that

$$I^{n+1} : x = I^n + y(I^n : x).$$

Applying this formula successively, we obtain $I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x)$. \square

Theorem 2.7. *Let A be a two-dimensional Buchsbaum ring with depth $A > 0$ and I an \mathfrak{m} -primary ideal. Let J be an arbitrary minimal reduction of I . Set $r = r_J(I)$. Then*

$$\text{reg } R(I) = \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}.$$

Proof. If $r = 0$, I is a parameter ideal. Since A is Buchsbaum, I is generated by a d -sequence. Hence $\text{reg } R(I) = 0$ by [29, Corollary 5.7]. In this case, the above formula is trivial. Therefore, we may assume that I is not a parameter ideal.

By the proof of Theorem 2.4, we have

$$\text{reg } R(I) = \min\{n \geq r \mid I^{n+1} : x = I^n\},$$

where x is an element of a superficial sequence x, y of I such that $J = (x, y)$. By Lemma 2.6, if $n \geq r$,

$$I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x) \subseteq I^n + I^{n-r}(\widetilde{I^{r+1}} : x).$$

By [22, Lemma 3.1 (5)], $\widetilde{I^{r+1}} : x = \widetilde{I}^r$. If $\widetilde{I}^r = I^n : I^{n-r}$, we have

$$I^n + I^{n-r}(\widetilde{I^{r+1}} : x) = I^n + I^{n-r}\widetilde{I}^r \subseteq I^n \subseteq I^{n+1} : x.$$

From this it follows that $I^{n+1} : x = I^n$. Thus,

$$\text{reg } R(I) \geq \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}.$$

To show the converse inequality we observe that for $n \geq r$,

$$\widetilde{I}^r \subseteq \widetilde{I}^n : I^{n-r} \subseteq \widetilde{I}^n : x^{n-r} = \widetilde{I}^r$$

by [22, Lemma 3.1 (5)]. From this it follows that $\widetilde{I}^r = \widetilde{I}^n : I^{n-r}$. If $\widetilde{I}^n = I^n$, we have $\widetilde{I}^r = I^n : I^{n-r}$. Therefore, using Theorem 2.4, we have

$$\begin{aligned} \text{reg } R(I) &= \min\{n \geq r \mid \widetilde{I}^n = I^n\} \\ &\leq \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}. \end{aligned}$$

\square

3. THE CONJECTURE OF EISENBUD AND ULRICH

Throughout this section let A be a finitely generated standard graded algebra over a field k with $\dim A > 0$. Let \mathfrak{m} be the maximal graded ideal of A and I an \mathfrak{m} -primary ideal generated by homogeneous elements of the same degree d , $d \geq 1$.

Motivated by the behavior of the function $\operatorname{reg} I^n$, Eisenbud and Ulrich conjectured that $\operatorname{reg} R(I) = \operatorname{reg} k[I_d]$, where $k[I_d]$ is the algebra generated by the elements of the component I_d of I [7, Conjecture 1.3]. Note that $k[I_d]$ is isomorphic to the fiber ring $F(I) = \bigoplus_{n \geq 0} I^n / \mathfrak{m} I^n$ because I is generated by elements of the same degree d .

The conjecture of Eisenbud and Ulrich is not true if one does not put further assumption on A . This follows from the following observation on graded Buchsbaum rings, where A is called a Buchsbaum ring if $A_{\mathfrak{m}}$ is a Buchsbaum ring.

Proposition 3.1. *Assume that $\operatorname{reg} R(Q) = \operatorname{reg} F(Q)$ for every parameter ideal Q generated by forms of the same degree in graded factor rings of A . Then A is a Buchsbaum ring.*

Proof. It is well known that every system of parameters is analytically independent. From this it follows that $F(Q)$ is isomorphic to a polynomial ring over k . Hence $\operatorname{reg} R(Q) = \operatorname{reg} F(Q) = 0$. By [29, Corollary 5.7], this implies that Q is generated by a d -sequence. In particular, every system of parameters of A , which consists of forms of the same degree, is a d -sequence.

Let x_1, \dots, x_s be a homogeneous system of parameters of degree 2 of A , $s = \dim A$. Applying the above fact to the factor ring $A/(x_1, \dots, x_i)$, $i < s$, we can deduce that every homogeneous system of parameters $x_1, \dots, x_i, y_1, \dots, y_{s-i}$ of A , where y_1, \dots, y_{s-i} are linear forms, is a d -sequence. By [26, Corollary 2.6], a local ring (B, \mathfrak{n}) is Buchsbaum if there exists a system of parameters x'_1, \dots, x'_s in \mathfrak{n}^2 , $s = \dim B$, and a generating set S for \mathfrak{n} such that $x'_1, \dots, x'_i, y'_1, \dots, y'_{s-i}$ is a d -sequence for every family y'_1, \dots, y'_{s-i} of $s-i$ elements of S , $i = 1, \dots, s$ (the term absolutely superficial sequence was used there for d -sequence). From this it follows that A is a Buchsbaum ring. \square

The following example shows that $\operatorname{reg} R(I)$ can be arbitrarily larger than $\operatorname{reg} F(I)$ even when I is a parameter ideal in an one-dimensional non-Buchsbaum ring.

Example 3.2. Let $A = k[x, y]/(x^t, xy^{t-1})$, $t \geq 2$. Then A is a non-Buchsbaum ring for $t \geq 3$. Let $I = yA$. It is clear that $\operatorname{reg} F(I) = 0$. Using Theorem 1.3, we have $\operatorname{reg} R(I) = \min\{n \geq 0 \mid y^{n+1}A \cap (0 : yA) = 0\} = t - 2$.

Let \mathfrak{n} denote the maximal graded ideal of $F(I)$. Since $F(I)$ is a standard graded algebra over k , $F(I) \cong G(\mathfrak{n})$. By Theorem 1.3,

$$\operatorname{reg} F(I) = \operatorname{reg} G(\mathfrak{n}) = \operatorname{reg} R(\mathfrak{n}),$$

and we can use a minimal reduction of \mathfrak{n} , to compute $\operatorname{reg} F(I)$.

A minimal reduction of \mathfrak{n} is just a parameter ideal of $F(I)$ generated by linear forms. In general, there is a natural correspondence between such parameter ideals and minimal reductions of I (see e.g. [31, Section 1.3]). In our setting, this correspondence can be formulated as follows.

In the following we will identify $F(I)$ with $k[I_d]$ and we will consider it as the graded subalgebra $\oplus_{n \geq 0} (I^n)_{nd}$ of $R(I)$. Let J be an arbitrary ideal generated by s forms $x_1, \dots, x_s \in I_d$, $s = \dim A$. Let \mathfrak{q} be the ideal generated by these forms in $F(I)$.

Lemma 3.3. *J is a minimal reduction of I if and only if \mathfrak{q} is a parameter ideal of $F(I)$. Moreover, $r_{\mathfrak{q}}(\mathfrak{n}) = r_J(I)$.*

If J is a minimal reduction of I , using the proof of Lemma 1.2 we can find a generating sequence x_1, \dots, x_s for J such that it is superficial for both J and \mathfrak{n} . Applying Theorem 1.3 and Lemma 3.3 we have

$$(1) \quad \text{reg } R(I) =$$

$$\min \{n \geq r_J(I) \mid I^{n+1} \cap [(x_1, \dots, x_{i-1}) : x_i] = (x_1, \dots, x_{i-1})I^n, i = 1, \dots, s\},$$

$$(2) \quad \text{reg } F(I) =$$

$$\min \{n \geq r_J(I) \mid \mathfrak{n}^{n+1} \cap [(x_1, \dots, x_{i-1})F(I) : x_i] = (x_1, \dots, x_{i-1})\mathfrak{n}^n, i = 1, \dots, s\}.$$

Since $\mathfrak{n}^{n+1} = \oplus_{t \geq n} (I^{t+1})_{(t+1)d}$, one can easily check that

$$(3) \quad \mathfrak{n}^{n+1} \cap [(x_1, \dots, x_{i-1})F(I) : x_i] = \bigoplus_{t \geq n} (I^{t+1} \cap [(x_1, \dots, x_{i-1}) : x_i])_{(t+1)d},$$

$$(4) \quad (x_1, \dots, x_{i-1})\mathfrak{n}^n = \bigoplus_{t \geq n} ((x_1, \dots, x_{i-1})I^t)_{(t+1)d}.$$

Therefore, one can use the formulas (1) and (2) to compare $\text{reg } R(I)$ and $\text{reg } F(I)$. In particular, one can easily see that

$$(5) \quad \text{reg } R(I) \geq \text{reg } F(I),$$

which was proved in [7, Section 1] by a different mean.

Proposition 3.4. *Let A be a Buchsbaum ring with $\dim A \geq 1$ and $\text{depth } G(I) \geq \dim A - 1$. Then $\text{reg } R(I) = \text{reg } F(I)$.*

Proof. By [28, Theorem 1.2], the assumption implies that $\text{reg } R(I) = r_J(I)$ for every minimal reduction J of I . Since $r_J(I) \leq \text{reg } F(I) \leq \text{reg } R(I)$ by (2) and (3), we obtain $\text{reg } R(I) = \text{reg } F(I)$. \square

The condition $\text{depth } G(I) \geq \dim A - 1$ is satisfied if $\dim A = 1$. Therefore, we have the following consequence.

Corollary 3.5. *Let A be an one-dimensional Buchsbaum ring. Then $\text{reg } R(I) = \text{reg } F(I)$.*

If $\dim A = 2$, we shall see that there is a formula for $\text{reg } F(I)$, which is similar to the formula for $\text{reg } R(I)$ in Theorem 2.4.

Let $s_{\text{in}}^*(I)$ denote the least integer m such that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$ for all $n \geq m$. It is clear that nd is the initial degree of the homogeneous elements of \tilde{I}^n . For this reason we call $s_{\text{in}}^*(I)$ the *initial Ratliff-Rush regularity*. Obviously, $s_{\text{in}}^*(I) \leq s^*(I)$.

Lemma 3.6. *Let I be a regular ideal. Then $s_{\text{in}}^*(I) \leq \max\{\text{reg } F(I), 1\}$.*

Proof. We have to show that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$ for $n \geq \text{reg } F(I)$. Since $\tilde{I}^n = \cup_{t \geq 0} I^{n+t} : I^t$, it suffices to show that $(I^{n+t} : I^t)_{nd} \subseteq (I^n)_{nd}$ for $n \geq \text{reg } F(I)$.

Let $x \in I_d$ be a superficial element of \mathfrak{n} . Since \mathfrak{n} is a regular ideal, x is a non-zerodivisor. By Lemma 1.4, $\mathfrak{n}^{n+1} \cap (x) = x\mathfrak{n}^n$ for $n \geq \text{reg } F(I)$. This implies $\mathfrak{n}^{n+1} : x = \mathfrak{n}^n$. From this it follows that $\mathfrak{n}^{n+t} : x^t = \mathfrak{n}^n$ for $n \geq \text{reg } F(I)$. Note that $(\mathfrak{n}^n)_d = (I^n)_{nd}$. Then

$$(I^{n+t} : I^t)_{nd} \subseteq (I^{n+t} : x^t)_{nd} = (\mathfrak{n}^{n+t} : x^t)_d = (\mathfrak{n}^n)_d = (I^n)_{nd}$$

for $n \geq \text{reg } F(I)$, which implies the conclusion. \square

Theorem 3.7. *Let A be a two-dimensional Buchsbaum ring with $\text{depth } A > 0$. Assume that I is not a parameter ideal. Then*

$$\text{reg } F(I) = \max\{r_J(I), s_{\text{in}}^*(I)\} = \min\{n \geq r_J(I) \mid (\tilde{I}^n)_{nd} = (I^n)_{nd}\},$$

where J is an arbitrary homogeneous minimal reduction of I .

Proof. Let $J = (x, y)$, $x, y \in I_d$. By (2) we have $\text{reg } F(I) \geq r_J(I)$. Since I is not a parameter ideal, \mathfrak{n} is not generated by two elements. From this it follows that the defining equations of $F(I)$ have degree > 1 . Hence $\text{reg } F(I) > 0$ [6]. By Lemma 3.6, this implies $\text{reg } F(I) \geq s_{\text{in}}^*(I)$. Thus, $\text{reg } F(I) \geq \max\{r_J(I), s_{\text{in}}^*(I)\}$. Since

$$\max\{r_J(I), s_{\text{in}}^*(I)\} \geq \min\{n \geq r_J(I) \mid (\tilde{I}^n)_{nd} = (I^n)_{nd}\},$$

it suffices to show that $\text{reg } F(I) \leq \min\{n \geq r_J(I) \mid (\tilde{I}^n)_{nd} = (I^n)_{nd}\}$.

By the proof of Lemma 2.6, we can choose x, y such that x, y is a superficial sequence for both I and \mathfrak{n} . Since I is regular ideal, x is a non-zerodivisor in A . Hence $0 : x = 0$ in $F(I)$. By (2) we have

$$\text{reg } F(I) = \min\{n \geq r_J(I) \mid \mathfrak{n}^{n+1} \cap (xF(I) : y) = x\mathfrak{n}^n\}.$$

By (3) and (4), $\mathfrak{n}^{n+1} \cap (xF(I) : y) = x\mathfrak{n}^n$ if $[I^{t+1} \cap (xA : y)]_{(t+1)d} = (xI^t)_{(t+1)d}$ for $t \geq n$. On the other hand, by the proof of Theorem 2.4, we have

$$I^{t+1} \cap (xA : y) = I^{t+1} \cap xA = x(I^{t+1} : x)$$

for $t \geq r_J(I)$. Therefore, we only need to show that $(I^{t+1} : x)_{td} = (I^t)_{td}$ for $t \geq n$ if $(\tilde{I}^n)_{nd} = (I^n)_{nd}$ for $n \geq r_J(I)$.

By Lemma 2.6, we have $I^{t+1} : x = I^t + y^{t-n}(I^{n+1} : x)$ for $t \geq n \geq r_J(I)$. Since $I^{n+1} : x \subseteq \widetilde{I^{n+1}} : x = \tilde{I}^n$ [22, Lemma 3.1(5)], $I^{t+1} : x \subseteq I^t + y^{t-n}\tilde{I}^n$. If $(\tilde{I}^n)_{nd} = (I^n)_{nd}$, we obtain

$$(I^{t+1} : x)_{td} \subseteq (I^t)_{td} + y^{t-n}(I^n)_{nd} = (I^t)_{td} \subseteq (I^{t+1} : x)_{td}.$$

From this it follows that $(I^{t+1} : x)_{td} = (I^t)_{td}$. \square

Corollary 3.8. *Let A be a two-dimensional Buchsbaum ring with $\text{depth } A > 0$. Let J be an arbitrary homogeneous minimal reduction of I . Then $\text{reg } R(I) = \text{reg } F(I)$ if and only if $\tilde{I}^n = I^n$ for the least integer $n \geq r_J(I)$ such that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$.*

Proof. If I is a parameter ideal, we have $\text{reg } R(I) = \text{reg } F(I) = 0$ by [29, Corollary 5.7] and $\tilde{I}^n = I^n$ for $n \geq 1$ by Corollary 2.2. If I is not a parameter ideal, the conclusion follows from Theorem 2.4 and Theorem 3.7. \square

4. MONOMIAL IDEALS IN TWO VARIABLES

In this section we will use the relationship between Castelnuovo-Mumford regularity and Ratliff-Rush closure to investigate the conjecture of Eisenbud and Ulrich for monomial ideals in two variables.

Let $A = k[x, y]$ be a polynomial ring over a field k , $\mathfrak{m} = (x, y)$ and I is an \mathfrak{m} -primary ideal generated by monomials of degree d , $d \geq 1$. In this case, I contains x^d, y^d and $J = (x^d, y^d)$ is a minimal reduction of I . It is well-known [24] and easy to see that

$$\tilde{I}^n = \bigcup_{t \geq 0} I^{n+t} : (x^{td}, y^{td}).$$

Lemma 4.1. *Let $I = (x^d, y^d) + (x^{d-i}y^i \mid a \leq i \leq b)$, where $a \leq b < d$ are given positive integers. Then $\tilde{I}^n = I^n$ for all $n \geq 1$.*

Proof. Let $x^i y^j$ be an arbitrary monomial of \tilde{I}^n . Then $x^{i+td} y^j \in I^{n+t}$ for some $t \geq 1$. Since I^{n+t} is generated by monomials of degree $(n+t)d$, $x^{i+td} y^j$ is divisible by a monomial $x^{(n+t)d-c} y^c \in I^{n+t}$. The divisibility implies $i+td \geq (n+t)d-c$ and $j \geq c$.

If $j < na$, then

$$(n+t)d-c \geq (n+t)d-j > (n+t)d+na=td+n(d-a).$$

Let $M = \{x^d, x^{d-a}y^a, x^{d-a-1}y^{a+1}, \dots, x^{d-b}y^b, y^d\}$ be the set of the monomial generators of I . Then $x^{(n+t)d-c}y^c$ is a product of $n+t$ monomials of M . Let s be the number of copies of x^d among these $n+t$ monomials of M . If $s < t$, we would have $(n+d)d-c \leq sd + (n+t-s)(d-a)$ because the exponent of x in each monomial in $M \setminus \{x^d\}$ is less or equal $d-a$. Since

$$sd + (n+t-s)(d-a) = td + n(d-a) - (t-s)a < td + n(d-a),$$

we would get $(n+d)d-c < td + n(d-a)$, a contradiction. Therefore, we must have $s \geq t$. From this it follows that $x^{nd-c}y^c = x^{(n+t)d-c}y^c/x^{td}$ is a product of n monomials in M . Hence $x^{nd-c}y^c \in I^n$. Since $x^i y^j$ is divisible by $x^{nd-c}y^c$, $x^i y^j \in I^n$.

By symmetry, if $i < n(d-b)$, we can also show that $x^i y^j \in I^n$.

Now, we may assume that $i \geq n(d-b)$ and $j \geq na$. Let Q denote the ideal generated by the monomials $x^{d-j}y^j$, $a \leq j \leq b$. It is clear that Q^n is generated by the monomials $x^{nd-j}y^j$, $na \leq j \leq nb$. If $j < nb$, $x^i y^j$ is divisible by $x^{nd-j}y^j$ because $i \geq nd-c \geq nd-j$. Therefore, $x^i y^j \in Q^n$. Since $Q \subset I$, we obtain $x^i y^j \in I^n$. If $j \geq nb$, then $x^i y^j$ is divisible by $x^{n(d-b)}y^{nb} = (x^{d-b}y^b)^n \in I^n$. Thus, we always have $x^i y^j \in I^n$. Therefore, we can conclude that $\tilde{I}^n = I^n$. \square

Theorem 4.2. *Let $I = (x^d, y^d) + (x^{d-i}y^i \mid a \leq i \leq b)$, where $a \leq b < d$ are given positive integers. Then*

$$\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$$

for any homogeneous minimal reduction J of I .

Proof. By Lemma 4.1, we have $s^*(I) = 1$. Since $s^*(I) \geq s_{\text{in}}^*(I) \geq 1$, we also have $s_{\text{in}}^*(I) = 1$. Applying Theorem 2.4 and Theorem 3.7, we obtain $\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$. \square

Theorem 4.2 gives a large class of monomial ideals in two variables for which the conjecture of Eisenbud and Ulrich holds. In particular, it contains the case I is an \mathfrak{m} -primary ideal generated by three monomials.

Corollary 4.3. *Assume that $I = (x^d, x^{d-a}y^a, y^d)$, $1 \leq a < d$. Then*

$$\operatorname{reg} R(I) = \operatorname{reg} F(I) = d/(a, d) - 1.$$

Proof. This is the case $a = b$ of Theorem 4.2. Therefore, $\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$. For $J = (x^d, y^d)$, it is easy to check that $r_J(I) = d/(a, d) - 1$. \square

Now we will present another large class of monomial ideals in two variables for which the conjecture of Eisenbud and Ulrich holds.

Theorem 4.4. *Let I be an ideal in $k[x, y]$ which is generated by monomials of degree $d \geq 2$. Assume that $x^d, x^{d-1}y, y^d \in I$. Then $\operatorname{reg} R(I) = \operatorname{reg} F(I)$.*

Proof. Let n be the least integer $n \geq r_J(I)$ such that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$. By Corollary 3.8, we only need to show that $\tilde{I}^n = (I^n)$.

Let $x^i y^j$ be an arbitrary monomial of \tilde{I}^n . Then $x^i y^{j+td} \in I^{n+t}$ for some $t \geq 1$. Since I^{n+t} is generated by monomials of degree $(n+t)d$, there exists a monomial $x^a y^{(n+t)d-a} \in I^{n+t}$ such that $x^i y^{j+td}$ is divisible by $x^a y^{(n+t)d-a}$. By the divisibility, we have $i \geq a$ and $j + td \geq (n+t)d - a$. For what follows see Figure I, where each node represents the vector of exponents of a monomial.

If $i \geq nd$, then $x^i y^j$ is divisible by $x^{nd} \in I^n$. If $i < nd$, then $a < nd$. We have

$$\begin{aligned} (x^a y^{nd-a}) x^{(nd-a)d} &= (x^{d-1} y)^{nd-a} x^{nd} \in I^{nd-a+n} \\ (x^a y^{nd-a}) y^{td} &= x^a y^{(n+t)d-a} \in I^{n+t}. \end{aligned}$$

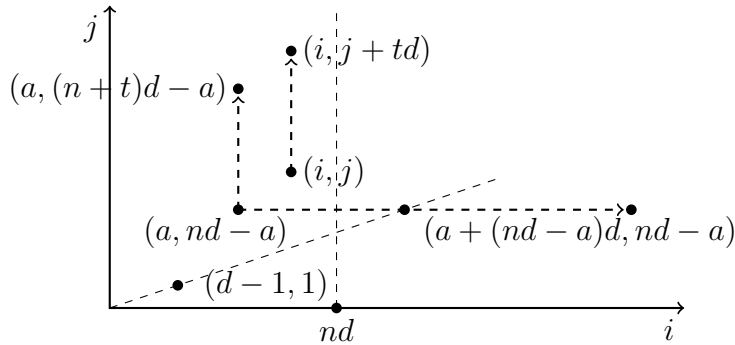


FIGURE 1.

Set $s = \max\{(nd-a), t\}$. Then $(x^a y^{nd-a}) x^{sd}, (x^a y^{nd-a}) y^{sd} \in I^{n+s}$. Hence $x^a y^{nd-a} \in I^{n+s} : (x^{sd}, y^{sd}) \subseteq \tilde{I}^n$. Since $(\tilde{I}^n)_{nd} \in (I^n)_{nd}$, $x^a y^{nd-a} \in I^n$. Since $i \geq a$ and $j \geq nd - a$, $x^i y^j$ is divisible by $x^a y^{nd-a}$. Thus, $x^i y^j \in I^n$. So we can conclude that $\tilde{I}^n = (I^n)$. \square

Actually, we obtain the above cases by translating everything in terms of lattice points. For every set $S \subseteq A$ we consider the set $E(Q)$ of the vectors of the exponents of the monomials in Q . We define the sum of two sets in \mathbb{N}^2 simply as the set of the corresponding sums of the elements of the given sets. A multiple of a set is thus a sum of copies of the given set.

Let E be the set of the vectors of the exponents of the monomial generators of I . Set $\mathbf{e}_1 = (d, 0)$ and $\mathbf{e}_2 = (0, d)$. For $J = (x^d, y^d)$ we have

$$r_J(I) = \min \{n \mid (n+1)E = \{\mathbf{e}_1, \mathbf{e}_2\} + nE\}.$$

Let E_n resp. F_n denote the set of the vectors $\mathbf{a} \in \mathbb{N}^2$ such that $\mathbf{a} + t\mathbf{e}_1, \mathbf{a} + t\mathbf{e}_2 \in (n+t)E$ resp. $\mathbf{a} + t\mathbf{e}_1, \mathbf{a} + t\mathbf{e}_2 \in (n+t)E + \mathbb{N}^2$ for some $t \geq 0$. Then $E((\tilde{I}^n)_{nd}) = E_n$ and $E(\tilde{I}^n) = F_n$. It is clear that $(\tilde{I}^n)_{nd} = (I^n)_{nd}$ resp. $\tilde{I}^n = I^n$ if and only if $E_n = nE$ resp. $F_n = nE + \mathbb{N}^2$.

Remark 4.5. The sets E_n have an interesting interpretation in the theory of affine semigroup rings. Let $S \subseteq \mathbb{N}^2$ denote the additive monoid generated by E . Then S is called an affine semigroup and $F(I)$ is the semigroup ring $k[S]$ of S . Set

$$S^* = \{\mathbf{a} \in \mathbb{N}^2 \mid \mathbf{a} + t\mathbf{e}_1, \mathbf{a} + t\mathbf{e}_2 \in S \text{ for some } t \geq 0\}.$$

By [27, Lemma 1.1], $k[S]$ is Cohen-Macaulay or Buchsbaum if and only if $S^* = S$ or $S^* + (S \setminus \{0\}) \subseteq S$, respectively. Actually, S^* is an additive semigroup such that $k[S^*]$ is Cohen-Macaulay, and we can prove that

$$H_{\mathfrak{n}}^1(k[S]) \cong k[S^*]/k[S],$$

where \mathfrak{n} denote the maximal graded ideal of $k[S]$ and $k[S^*]/k[S]$ is the vector space spanned by the elements of $S^* \setminus S$. It is easy to see that $S^* \setminus S = \cup_{n \geq 1} (E_n \setminus nE)$ and that the n -th graded component of $H_{\mathfrak{n}}^1(k[S])$ is a vector space spanned by the elements of the set $E_n \setminus nE$.

Now we will give an example such that $\text{reg } R(I) = s^*(I) = r_J(I)$ for $J = (x^d, y^d)$, but $s^*(I) > r_{J'}(I)$ for another minimal reduction J' of I .

Example 4.6. Let $I = (x^7, x^6y, x^2y^5, y^7)$. First, we will show that $\text{reg } R(I) = r_J(I)$ for $J = (x^7, y^7)$. By Theorem 4.4 we know that $\text{reg } R(I) = \text{reg } F(I)$. Since $\text{reg } F(I) \geq r_J(I)$ by (2), it suffices to show that $\text{reg } F(I) = r_J(I)$. It is easy to check that $r_J(I) = 4$ and $(\tilde{I}^4)_{28} = (I^4)_{28}$. By Theorem 3.7, this implies $\text{reg } F(I) = 4$. On the other hand, it is shown in [15, Example 3.1] that $r_{J'}(I) \leq 3$ for $J' = (x^7, x^6y + y^7)$. Hence the reduction numbers of I depend on the choice of the minimal reductions. By Theorem 2.4, this implies $\text{reg } R(I) = s^*(I) = r_J(I)$.

The following example shows that we may have $\text{reg } R(I) = s^*(I) > r_J(I)$, where $J = (x^d, y^d)$.

Example 4.7. Let $I = (x^{17-i}y^i \mid i = 0, 1, 3, 5, 13, 14, 16, 17)$. By [12, Example 3.2], we have $\text{reg } F(I) = 4 > r_J(I) = 3$, where $J = (x^{17}, y^{17})$. By Theorem 4.4 we have $\text{reg } R(I) = \text{reg } F(I) = 4$. Hence, Theorem 2.4 implies $\text{reg } R(I) = s^*(I) > r_J(I)$.

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